Tests of structural changes in conditional distributions with unknown changepoints

Dominique Guegan, Philippe de Peretti

Documents de Travail du Centre d'Economie de la Sorbonne

CES Working Papers

Tests of Structural Changes in Conditional Distributions with Unknown Changepoints

Dominique GUEGAN, Philippe de PERETTI

2011.42

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

Tests of Structural Changes in Conditional Distributions with Unknown Changepoints

Dominique Guégan* Philippe de Peretti†

July 18, 2011

Abstract

This paper focuses on a procedure to test for structural changes in the first two moments of a time series, when no information about the process driving the breaks is available. To approximate the process, an orthogonal Bernstein polynomial is used, and testing for the null is achieved either by using an $AICu$ information criterion, or a restriction test. The procedure covers both the pure discrete structural change and the continuous changes models. Running Monte-Carlo simulations, we show that the test has power against various alternatives.

Keywords: Structural Changes ; Bernstein polynomial ; $AICu$

1 Introduction

This paper deals with models of the form:

$$A(L)y_t = c_t \tag{1}$$

where: $A(L) = 1 - \rho_1 L - \rho_2 L^2 - ... - \rho_p L^p$,

c_t is either defined as $c_t = f(t) + \varepsilon_t$ or $c_t = f(t)\varepsilon_t$, where in both cases $f(t)$ is

*Université Paris1 Panthéon-Sorbonne, 106-112 Bd de l'Hôpital, 75013 Paris, France. Dguegan@univ-paris1.fr

†Université Paris1 Panthéon-Sorbonne, 106-112 Bd de l'Hôpital, 75013 Paris, France. philippe.de-peretti@univ-paris1.fr

an unknown, possibly time-varying signal, thus inducing heterogeneity in one of the two moments of the conditional distribution, ε_t is an iid term.
For instance, assume the simple case where $f(t)$ is defined as a step-function for the mean:

$$f(t) = \begin{cases} \alpha_1 \text{ if } t \leq t_0 \\ \alpha_2, \text{ otherwise} \end{cases} \tag{2}$$

with $\alpha_1 \neq \alpha_2$ and $t_0 \in (\lambda_1 T, T(1-\lambda_2))$.
Perron (2005) stresses the importance of testing for structural changes. On the one hand, structural changes are a source of global non-stationarity (Granger and Starica [2005], Guégan [2010]) and then of parameter instability, and on the other hand, ignoring structural changes may lead to erroneous statistical inference in tests for stationarity (Perron [1989]), and for long memory (Diebold and Inoue [2001], Charffedine and Guégan [2011]). Thus, testing for time-heterogeneity of moments in a time series is a prior to modelling.

Several procedures have been developed to test for multiple changes when the number of changes is unknown. For instance, Bai and Perron (1998) and Andrews, Lee and Ploberger (1996) have suggested a sequential procedure, while Bai and Perron (1998) have also introduced the so-called double-maximum test. In a recent contribution, Heracleous, Koutris and Spanos (2004) have pointed out that such procedures may not have power against continuous changes. Instead they suggest computing rolling moments on (filtered) series, and then testing for time heterogeneity of these moments using an orthogonal polynomial, this latter capturing movements in moments.

In this paper, we present an alternative procedure inspired of that of Heracleous, Koutris and Spanos (2004). The procedure tests for the null of no structural change in the first two moments of a conditional distribution of a time series against a pure discrete break model, or continuous changes in moments. Compared with the above literature, the suggested test differs in several ways: i) The procedure requires no estimation of the breaks, and then of $f(t)$, ii) The procedure is not sequential, iii) The procedure does not use rolling windows estimators for the moments, thus avoiding the difficult choice of choosing a window, iv) At last the procedure has not the nuisance parameter problem under the alternative.

Our aim is to estimate model (1), by approximating the unknown function $f(t)$ by an orthogonal polynomial, here a Bernstein one. With k the degree of the polynomial, the test of no-structural breaks therefore amounts to testing $k = 0$ (constant signal) against $k > 0$. To perform such a task, we use two statistical strategies. The first one consists in using an information criterion to select the optimal model. We jointly select the order p and the degree k using the $AICu$ criterion (McQuarrie and Tsai [1988]). Indeed, this criterion ensures an optimal trade-off between smoothing and fitting. Since the $AICu$ is an 'all or nothing' decision rule, we also focus on a two-step strategy consisting in i) Selecting the optimal model using the $AICu$ and then if $k > 0$, ii) Using a restriction test to test $k = 0$ against $k > 0$.

This note is organized as follows. Section 2 presents the test, Section 3 implements Monte-Carlo simulations, and Section 4 concludes.

2 A test of no structural change

For $\{y_t\}_{t=1}^{T}$ where y_t is real-valued, we define the following Data Generating Process:

$$y_t = \sum_{i=1}^{p} \rho_i y_{t-i} + c_t \tag{3}$$

Suppose we are interested in testing for first-order time homogeneity: $H_0^1 : c_t = \alpha_1 + \varepsilon_t$, and conditional on H_0^1 true, for second-order time homogeneity: $H_0^2 : c_t = \alpha_2 \varepsilon_t$, Under H_0^1 and H_0^2, (3) can be re-written as:

$$y_t = \sum_{i=1}^{p} \rho_i y_{t-i} + c + \varepsilon_t \tag{4}$$

where: ε_t is an iid noise.

Since $f(t)$ is generally unknown in empirical work, we approximate the unknown signal by an orthogonal Bernstein polynomial. The unconstrained model is thus given by (5) for the mean:

$$
\begin{aligned}
y_t &= \sum_{i=1}^{p} \rho_i y_{t-i} + f(t) + \varepsilon_t && (5)\\
&= \sum_{i=1}^{p} \rho_i y_{t-i} + \sum_{i=0}^{k} \beta_i \binom{k}{i} (\frac{t}{T})^i (1-\frac{t}{T})^{k-i} + \varepsilon_t && (6)
\end{aligned}
$$

and (7) for the variance under H_0^1 true.

$$
\varepsilon_t^2 = \sum_{i=0}^{k} \beta_i \binom{k}{i} (\frac{t}{T})^i (1-\frac{t}{T})^{k-i} + \nu_t \qquad (7)
$$

where: ε_t^2 are the squared residuals of model (4), and ν_t is an iid noise

It is straightforward to see that in models (6) and (7), no-structural change in the conditional distribution of y_t implies $k = 0$, corresponding to a constant signal. Thus, testing for the null amounts to testing: $H_0^i : k = 0$ against $k > 0$, $i = 1, 2$.

Two testing strategies are used:

- Select the adequate model by minimizing an information criterion. Note that since we want to extract a signal, a classical mean square error (MSE) minimization criterion will be inadequate, resulting in overweighting the fit. This leads to use a penalized MSE. For an optimal trade-off between fitting and smoothing, we use the $AICu$ criterion, introduced by McQuarrie and Tsai (1988), and given by:

  $$
  AICu = \log(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}(T-p-k)^{-1}) + 2(p+k+1)(p-k-2)^{-1} \qquad (8)
  $$

  for the mean, and:

  $$
  AICu = \log(\boldsymbol{\nu}'\boldsymbol{\nu}(T-k)^{-1}) + 2(k+1)(k-2)^{-1} \qquad (9)
  $$

  for the variance under H_0^i true.

- Select the adequate model by minimizing the information criterion, if $k > 0$, test $k = 0$ against $k > 0$ using a restriction test. A typical procedure is then to use tests in a non-nested environment. In what follows, for instance for the mean, we estimate (10)

$$y_t = \sum_{i=1}^{p} \rho_i y_{t-i} + \beta_0 + \sum_{i=1}^{k} \beta_i \binom{k}{i} (\frac{t}{T})^i (1 - \frac{t}{T})^{k-i} + \varepsilon_t \quad (10)$$

and test $H_0^1 : \beta_1 = \beta_2 ... = \beta_k$ using a standard Ftest[1].

For the variance, under H_0^1 true, we estimate (11):

$$\varepsilon_t^2 = \alpha_0 + \sum i = 1^k \alpha_i \binom{k}{i} (\frac{t}{T})^i (1 - \frac{t}{T})^{k-i} + \nu_t \quad (11)$$

and test $H_0^2 : \alpha_1 = \alpha_2 ... = \alpha_k$.

We next turn to Monte-Carlo simulations.

3 Monte Carlo Simulations

In this section, we perform Monte-Carlo simulations to estimate the size and power of the test for various sample sizes and under different kinds of structural changes. The five cases for $c_t = f(t) + \varepsilon_t$, $\varepsilon_t \sim N(0,1)$ are:

- $f(t) = 0$, (iid case),
- $f(t) = 0$ for $t \leq t_0$ and $f(t) = 1$ otherwise and t_0 is randomly drawn in $(T/4, 3T/4)$ at each iteration (mean break),
- $f(t) = (1 + 2t/T)$ (mean trend),
- $f(t) = f(t-1) + \upsilon_t, \upsilon_t \sim N(0,1), f(0) = 0$ (stochastic trend),
- $y_t = f(t) + \varepsilon_t$, $f(t) = f(t-1) + \frac{\varepsilon_t^2}{\gamma + \varepsilon_t^2}, f(0) = 0$ (stop-break model).

Table 1 returns the results of the simulations when one bases the testing strategy on the simple $AICu$ criterion. The iid case is used to study the empirical size of the procedure, which is computed as $1 - P(k = 0)$ or $P(k = 1) \cup P(k > 1)$. The four other cases are used to compute the empirical

[1]For the test, Hetetoscedastic and Autocorrelation Consistent (HAC) matrices are used.

Table 1: $AICu$ based criterion for five models, the last four ones exhibiting ruptures in mean

iid case: H_0 true					
	$T=50$	$T=100$	$T=150$	$T=200$	$T=500$
$P(k=0)$	0.813	0.831	0.860	0.872	0.882
$P(k=1)$	0.102	0.090	0.088	0.066	0.075
$P(k>1)$	0.085	0.079	0.052	0.062	0.043
Single discrete break in mean: H_0 false					
	$T=50$	$T=100$	$T=150$	$T=200$	$T=500$
$P(k=0)$	0.125	0.018	0.006	0.001	0.000
$P(k=1)$	0.400	0.329	0.187	0.109	0.006
$P(k>1)$	0.475	0.653	0.807	0.890	0.994
Linear trend in mean: H_0 false					
	$T=50$	$T=100$	$T=150$	$T=200$	$T=500$
$P(k=0)$	0.047	0.000	0.000	0.000	0.000
$P(k=1)$	0.729	0.849	0.841	0.865	0.878
$P(k>1)$	0.224	0.160	0.159	0.135	0.122
Stochastic trend in mean: H_0 false					
	$T=50$	$T=100$	$T=150$	$T=200$	$T=500$
$P(k=0)$	0.229	0.244	0.252	0.233	0.228
$P(k=1)$	0.161	0.182	0.182	0.167	0.158
$P(k>1)$	0.610	0.574	0.566	0.600	0.614
Stop-break mode: H_0 false					
	$T=50$	$T=100$	$T=150$	$T=200$	$T=500$
$P(k=0)$	0.156	0.059	0.049	0.064	0.000
$P(k=1)$	0.202	0.077	0.055	0.078	0.000
$P(k>1)$	0.642	0.864	0.896	0.858	1.000

Note 1: The iid case returns the size of the procedure, given by $1-P(k=0)$. Ideally it should be close to 0

Note 2: The other four cases return the power of the procedure, given by $P(k=1)\cup P(k>1)$. Ideally it should be close to 1

power, i.e. $P(k=1)\cup P(k>1)$. Using the $AICu$ criterion returns a low empirical size, especially for small sample size ($T=50$), not exceeding 0.187. Focusing on the power, results are twofold. For the stop-break model (Engle and Smith [1999]), the linear trend in mean, and the single discrete break in mean models,

Table 2: Size and power of restriction tests at 4 nominal sizes for five models, the last four ones exhibiting ruptures in mean.

iid case: H_0 true					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.045	0.039	0.024	0.020	0.011
0.05	0.105	0.086	0.063	0.051	0.045
0.10	0.139	0.120	0.093	0.088	0.070
0.15	0.166	0.148	0.111	0.105	0.092
Single discrete break in mean: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.343	0.716	0.910	0.944	1.000
0.05	0.671	0.925	0.970	0.982	1.000
0.10	0.810	0.967	0.989	0.996	1.000
0.15	0.848	0.978	0.992	0.998	1.000
Linear trend in mean: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.489	0.918	0.995	1.000	1.000
0.05	0.788	0.986	1.000	1.000	1.000
0.10	0.897	0.994	1.000	1.000	1.000
0.15	0.938	0.998	1.000	1.000	1.000
Stochastic trend in mean: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.512	0.467	0.442	0.442	0.432
0.05	0.689	0.652	0.639	0.638	0.654
0.10	0.739	0.718	0.695	0.717	0.729
0.15	0.763	0.741	0.736	0.751	0.749
Stop-break model: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.670	0.871	0.880	0.860	0.999
0.05	0.792	0.913	0.936	0.914	1.000
0.10	0.817	0.933	0.946	0.925	1.000
0.15	0.831	0.936	0.947	0.932	1.000

Note 1: The iid case returns the size of the procedure. Ideally it should be close to the nominal one

Note 2: The four other cases return the power of the procedure. Ideally it should be close to 1

Table 3: $AICu$ based criterion for four models, the last three ones exhibiting ruptures in variance

iid case: H_0 true					
	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
$P(k = 0)$	0.932	0.898	0.900	0.897	0.888
$P(k = 1)$	0.043	0.073	0.074	0.069	0.074
$P(k > 1)$	0.023	0.029	0.026	0.034	0.038
Single discrete break in variance: H_0 false					
	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
$P(k = 0)$	0.391	0.085	0.001	0.001	0.000
$P(k = 1)$	0.476	0.635	0.462	0.109	0.145
$P(k > 1)$	0.133	0.410	0.537	0.890	0.855
Linear trend in variance: H_0 false					
	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
$P(k = 0)$	0.652	0.186	0.111	0.048	0.000
$P(k = 1)$	0.288	0.154	0.783	0.838	0.862
$P(k > 1)$	0.060	0.660	0.106	0.114	0.132
Stochastic trend in variance: H_0 false					
	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
$P(k = 0)$	0.380	0.186	0.160	0.110	0.107
$P(k = 1)$	0.306	0.154	0.101	0.088	0.032
$P(k > 1)$	0.314	0.660	0.739	0.802	0.861

Note 1: The iid case returns the size of the procedure, given by $1 - P(k = 0)$. Ideally it should be close to 0

Note 2: The three other cases return the power of the procedure, given by $P(k = 1) \cup P(k > 1)$. Ideally it should be close to 1

the power ranges from 0.844 to 0.953 for $T = 50$. For the stochastic trend in mean model, the power is lower: ranging form 0.771 to 0.748 according to the sample size. Hence, results remain within an acceptable range.

Focusing now on restriction tests, as presented in Table 2, at the 5% nominal size, the empirical sizes range from 0.105 ($T = 50$) to 0.045 ($T = 500$). Also, as mentioned above, for the stop-break, the linear trend in mean, and the single discrete break in mean models, the type II error is close to the nominal size, except for $T = 50$. When the data contain a stochastic trend, the power ranges

Table 4: Size and power of restriction tests at 4 nominal sizes for four models, the last three ones exhibiting ruptures in mean

iid case: H_0 true					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.007	0.007	0.013	0.009	0.012
0.05	0.024	0.026	0.030	0.031	0.038
0.10	0.043	0.052	0.055	0.060	0.061
0.15	0.054	0.070	0.076	0.080	0.082

Single discrete break in variance: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.109	0.459	0.768	0.913	1.000
0.05	0.305	0.729	0.947	0.985	1.000
0.10	0.447	0.839	0.980	0.995	1.000
0.15	0.538	0.880	0.988	0.998	1.000

Linear trend in variance: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.048	0.194	0.399	0.534	0.994
0.05	0.148	0.415	0.659	0.807	1.000
0.10	0.212	0.536	0.786	0.887	1.000
0.15	0.281	0.608	0.835	0.917	1.000

Stochastic trend in variance: H_0 false					
size	$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$
0.01	0.226	0.577	0.666	0.761	0.849
0.05	0.407	0.707	0.765	0.825	0.864
0.10	0.502	0.776	0.795	0.853	0.873
0.15	0.577	0.792	0.818	0.871	0.882

Note 1: The iid case returns the size of the procedure. Ideally it should be close to the nominal one

Note 2: The three other cases return the power of the procedure. Ideally it should be close to 1

form 0.639 to 0.689, at 5% suggesting using a higher threshold in empirical work.

Turning now to structural breaks in variances, four cases are considered:

- $f(t) = 1$ and thus $c_t = \varepsilon_t, \varepsilon_t \sim N(0,1)$ (iid case),
- $f(t) = 1$ for $t \leq t_0$ and $\sqrt{f(t)} = 2$ otherwise, $\varepsilon_t \sim N(0,1)$, and t_0 is randomly drawn in $(T/4, 3T/4)$ at each iteration (variance break),

- $f(t) = (1 + 2t/T)$, $\varepsilon_t \sim N(0,1)$ (variance trend),
- $f(t) = \exp(h_t/2)$, $h_t = h_{t-1} + \epsilon_t$, $\upsilon_t \sim N(0,1)$, $\varepsilon_t \sim N(0,1)$ (stochastic volatility model)

Table 3 presents the size and power of the procedure based on the $AICu$ decision rule. Clearly, the size is low, but unexpectedly doesn't decrease with the sample size. Considering the power, it is quite low for $T = 50$, especially when the variance moves according to a linear trend and generally for all considered models. It is nevertheless acceptable for sample sizes ranging from $T = 100$ to $T = 500$. Turning now to restriction tests, Table 4, it can be seen that the empirical size is less than the 5% nominal one. Focusing at last on the type II error; it appears that the test has power against the three models, only for sample sizes more or less than $T = 100$ ($T = 150$ for linear trend in variance). In all cases under the alternative, the test has low power for small sample sizes ($T = 50$).

4 Conclusion

In this note, we have introduced a procedure to test for the null of no structural change in the first two moments of a conditional distribution of a time series. The procedure uses Bernstein polynomials to extract the (noisy) signal and has not the nuisance parameter problem under the alternative. Two tests are proposed, a test based on the simple $AICu$ criterion, and a restriction one. Monte-Carlo simulations suggest that the test is powerful and can be used in empirical work. Moreover, the procedure could be used as a general misspecification test.

References

[1] Andrews, W.K., Lee, I. & W. Ploberger (1996), Optimal Change Point Tests for Normal Linear Regression, *Journal of Econometrics* 70, p. 9-38.

[2] Bai, J. & P. Perron (1998), Estimating and testing linear models with multiple strucural changes, *Econometrica* 66, p.47-78.

[3] Charfeddine, L. & D. Guégan (2011), Which is the best model for the US inflation rate : A structural changes model or a long memory process ? to appear in *Journal of Applied Econometrics.*

[4] Diebold, F.X. & A. Inoue (2001), Long memory and regime switching, *Journal of Econometrics* 105, p. 131-159.

[5] Engle, R.F. & A.D. Smith (1999), Stochastic permanent breaks, *The Review of Economics and Statistics* 81, p.553-574.

[6] Granger, C,. & C. Starica (2005), Nonstationarities in Stock Returns, *The Review of Economics and Statistics* 87, p. 503-522.

[7] Guégan, D. (2010), Non-stationary samples and meta-distribution. In: Basu A, Samanta T, SenGupta A, (eds.) ISI Platinium Jubilee Volume: Statistical sciences and interdisciplinary research, ICSPRAR World Scientific Review, in press.

[8] Heracleous, M.S., Koutris, A. & A. Spanos (2008), Testing for nonstationarity using maximum entropy resampling: A misspecification testing perspective, *Econometric Reviews* 27, p. 363-384.

[9] McQuarrie, A.D.R. & C.-L. Tsai (1998), Regression and time series model selection, World Scientific.

[10] Perron, P. (1989), The Great Crash, the oil price shock and the unit root hypothesis. *Econometrica* 57, p. 1361-1401.

[11] Perron, P. (2005), Dealing with Structural Breaks, *in* Palgrave Handbook of Econometrics, Vol. 1: Econometric Theory.